Philosophy of Mathematics

Semantic Realism: Why Mathematicians Mean What They Say

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ABSTRACT: I argue that if we distinguish between ontological realism and semantic realism, then we no longer have to choose between platonism and formalism. If we take category theory as the language of mathematics, then a linguistic analysis of the content and structure of what we say in and about mathematical theories allows us to justify the inclusion of mathematical concepts and theories as legitimate objects of philosophical study. Insofar as this analysis relies on a distinction between ontological and semantic realism, it relies also on an implicit distinction between mathematics as a descriptive science and mathematics as a descriptive discourse. It is this latter distinction which gives rise to the tension between the mathematician qua philosopher. In conclusion, I argue that the tensions between formalism and platonism, indeed between mathematician and philosopher, arise because of an assumption that there is an analogy between mathematical talk and talk in the physical sciences.

In this paper I argue that if we distinguish between ontological realism (the claim that mathematical objects exist independently of their linguistic expression) and semantic realism (the claim that mathematical statements which talk about mathematical objects are meaningful), then we no longer have to choose between platonism and formalism. If we take category theory as the language of mathematics, then a linguistic analysis of the content and structure of what we say in, and about, mathematical theories allows us to justify the inclusion of mathematical concepts and theories as legitimate objects of philosophical study. Insofar as this analysis relies on a distinction between ontological and semantic realism, it relies also on an implicit distinction between mathematics as a descriptive science (the view that mathematics is about mathematical objects) and mathematics as a descriptive discourse (the view that mathematics talks about mathematical objects). It is this latter distinction, I argue, which gives rise to the tension between the mathematician qua mathematician and the mathematician qua philosopher. When the mathematician claims that a mathematical object exists he intends to make a semantic claim. On the other hand, when the physical scientist claims that an object exists he intends to make an ontological claim. Thus, when the philosopher comes to analyze "existence" claims, he must be careful to distinguish between these intentions. In
conclusion, I argue that the tensions between formalism and platonism, indeed between mathematician and philosopher, arise because of an assumption that there is an analogy between mathematical talk and talk in the physical sciences.

Dieudonné characterizes the mathematician as follows:

we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism and say "mathematics is just a combination of meaningless symbols" . . . Finally we are left in peace to go back to our mathematics, with the feeling each mathematician has that he is working on something real. (Dieudonné in Davis and Hersh, [1981], p. 321)

Given this description, one may ask: Do mathematicians by nature have multiple-personalities or do philosophers make them crazy? Davis and Hersh would have us believe the former:

the typical mathematician is a [realist] on weekdays and a formalist on Sundays. That is, when he is doing mathematics he is convinced that he is dealing with an objective reality . . . But then, when challenged to give a philosophical account of this reality, he finds it easiest to pretend that he does not believe in it after all. (Davis and Hersh, [1981], p.321)

What explains the mathematician's double stance, according to Davis and Hersh, is his double roles. The mathematician qua mathematician is a realist, the mathematician qua philosopher is a formalist. Benacerraf, on the other hand, holds that the mathematician is driven to this personality shift by two conflicting philosophical demands, namely:

(1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of language, and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology. (Benacerraf in Hart, [1996], p. 14)

Thus, for Benacerraf it is not the double role of the mathematician that accounts for his differing views. Rather it is the philosophers' double demand for a homogeneous semantics and a reasonable (causal) epistemology.

In this paper I argue that if we distinguish between ontological realism (the claim that mathematical objects exist independently of their linguistic expression) and semantic realism (the claim that mathematical statements which talk about mathematical objects are meaningful), then we no longer have to chose between platonism and formalism, or worry about whether our semantics and epistemology matches that of physical discourse. If we construe category theory as the language of mathematics, then a linguistic analysis of the content and structure of what we say allows us to justify the inclusion of mathematical concepts and theories as legitimate objects of philosophical study. Specifically, such an analysis permits us to justify the claim that mathematical objects exist independently of us but, at the same time, depend on the structure of a given category and the content of a given theory.

Inasmuch as this analysis relies on a distinction between ontological and semantic realism, it relies also on an implicit distinction between mathematics as a descriptive science (the view that mathematics is about objects) and mathematics as a descriptive discourse (the view that mathematics talks about objects). It is this latter distinction, I argue, which gives rise to the tension between that mathematician qua mathematician and the mathematician...
qua philosopher. When the mathematician claims that a mathematical object exists he intends to make a semantic claim: he intends to say that the statement in which its concept occurs meaningfully refers. When the physical scientist claims that a physical object exists he intends to make an ontological claim: he intends to say that the concept is meaningful in virtue of its reference to an independently existing object. Thus, when the philosopher comes to analyze "existence" claims, he must be careful to distinguish between the intentions that motivate such claims.

1. The Semantic Tradition

In this section I argue that we have yet to appreciate the significance of what Coffa, [1991], has termed "the semantic tradition": we continue to assume that the conditions for meaningful and true assertions must be grounded in either what we can know or what exists. As a consequence, we continue to read those philosophical issues that relate to the semantics of mathematical discourse as strictly epistemological (either naturalized or psychologized) or strictly ontological (either physicalized or platonized). In contrast to such readings, the semantic tradition is characterized by the attempt to justify the inclusion of concepts as legitimate objects of philosophical study by "offering an analysis of the content and structure of what we say, as opposed to considering the psychological (or causal) acts by which we come to say it". (Coffa, [1991], p.1)

The semantic tradition began as a reaction to the psychologistic interpretation of Kant. In particular, philosophers and mathematicians belonging to this tradition sought to maintain the a priori character of mathematics while at the same time avoiding the seemingly necessary appeal to pure intuitions. As Coffa notes,

"the semantic tradition consisted of those who believed in the a priori but not in the constitutive powers of the mind . . . the root of all idealist confusion lay in misunderstandings concerning matters of meaning. Semanticists are easily detected: They devote an uncommon amount of attention to concepts, propositions, senses . . ." (Coffa, [1991], p.1)

Bolzano, in contrast to Kant, sought to account for the nature of both mathematical and philosophical concepts not in terms of the conditions for the possibility of experience, but in terms of the conditions for the possibility of saying. He sought to make a study "not of the transcendental considerations but of what we say and its laws . . . of semantics, not metaphysics or ontology". (Coffa, [1991], p. 23)

Bolzano's aim, then, was to show that like philosophical truths, "mathematical truths can and must be proven from the mere [the analysis of] concepts". (Bolzano quoted in Coffa, [1991], p.22) He did this by demonstrating how "mathematical rigor [could be] both an epistemological as well as a semantic notion". (Coffa, [1991], p. 26) This latter point is crucial for understanding the relevance of the semantic tradition, for as Coffa tells us

[it is widely thought that the principle inspiring such reconstructive efforts was epistemological, that they were basically a search for certainty. This is a serious error . . . a no less important purpose was the clarification of what was being said. (Coffa, [1991], p. 26)"

Underlying Bolzano's demonstration of the dual character of mathematical rigor was the distinction between what he termed 'subjective' and 'objective' representations. The former being the mental states of the soul and the latter the intersubjective content of the subjective representation. Meaning, then, relates not to the subjective representation but rather relates
to the intersubjective content and as such is in no need of assistance from intuitions (either
empirical or pure). Thus, what Bolzano did, more that bringing rigor to mathematics, was
effectively extrude mathematical concepts from the mind and place them firmly on
semantic soil.

This separation of semantics from psychology likewise enabled a distinction between both
ontological realism and idealism on the one hand and, what I have termed, semantic realism
on the other. Objective representations are not ontological entities, "they subsist not indeed
as something existing, but as a certain something . . .". (Coffa, [1991], p. 30) Nor are they
psychological entities:

[they] are the substance (Stoff) or [intersubjective] content of subjective representation. There
being in no way depends on the existence of subjective acts, just as the meaningfulness of
expressions in no way depends on anybody's having the appropriate meanings in mind; and like
meanings, there is only one for each linguistic unit. (Coffa, [1991], p. 30)

Unfortunately, this latter demand (that there be a isomorphism between the intersubjective
content of subjective representations and the meaning of objective representations) was
thought by Bolzano to be satisfied by the assumption that there is an isomorphism between
words (in natural language) and objective representations, i.e., by the assumption that "there
is only one objective representation designated by the word." (Bolzano quoted in Coffa,
[1991], p. 30)

Frege overcame this problem first, in the Begriffsschrift, by creating a language of concepts,
as opposed to relying on natural language to give names to objective representations.
Second, to Bolzano's demand that we "always separate sharply the psychological from the
logical, the subjective from the objective", he added, in his Die Grundlagen der Arithmetik,
that we "never ask for the meaning of a word in isolation, but only in the context of a
proposition". (Frege, [1978], p.x.) Given these proscriptions, arithmetic could proceed, as
philosophy, through the analysis of the content and structure of its concepts, modulo the
additional demand that, when writ in the language of the Begriffsschrift, such an analysis
need only consider what can be said of the relation between concepts and objects.
Regrettably, while Frege's context principle solved Bolzano's semantic problem at the level
of concepts, what Russell's paradox shows is that it does not solve the problem at the level
of objects. Basic Law V has an existential content which is conceivably false, i.e., which
cannot be justified from within Frege's language, and therefore he could not conclude that
arithmetic was analytic.

It is clear that both Bolzano and Frege exemplify the semantic tradition. What is not clear,
however, is whether the problems they encountered imply the demise of the semantic
approach itself. If we are to take history as our guide, then it appears as though we ought to
conclude that we must supplement our semantic analysis with the assumption that either
logic or reality provides a basis for what we say. Does this historical conclusion truly
represent all of our options? To answer this question we must first recognize what
assumption is necessary for such a drastic conclusion, viz., that the only way I can use a
statement to talk about objects is if the statement is about objects. What this assumption
does, however, is conflate the linguistic claim that meaningful statements talk about objects,
with the ontological claim that meaningful statements are about objects. It is this conflation,
I argue, that leads to the lack of distinction between ontological realism and semantic
realism.
While semantic philosophers realized that sense need not be constructed out of psychological intuitions, philosophers following them attempted to legitimize mathematical and/or theoretical concepts by offering a logical analysis not of what we say but by, in effect, reducing them to logical and/or empirical 'atoms of meaning'. Indeed it is the conflation of linguistic and ontological claims that lead philosophers, like Wittgenstein and Quine, to believe that unless we restrict ourselves to the world of facts or have access to 'real essences', as opposed to their nominalistic counterparts, then what we say will be so intimately bound to the conventions we choose, that any semantic analysis of it will show nothing. Thus, it is here that we must apply the lessons of the semantic tradition, before we take our cue from the philosophical history. For, as Coffa warns:

> few things have proved more difficult to achieve in the development of semantics than recognition of the fact that between our subjective representations and the world of things we talk about, there is a third element, what we say . . . many of the best philosophical minds . . . were [and are!] unable to understand that what we say, sense, cannot be constituted either from psychological content or from real-world [or logical] correlates of our representation . . .

(Coffa, [1991], p.77)

### 2. An Analysis of Discourses

The questions that I want now to consider are: Is conventionalism a necessary consequence of a linguistic analysis of mathematical discourse? and Why does a linguistic analysis not suffice for physical discourse? In this section I argue that if we take category theory as the language of mathematics, then we can not only justify the inclusion of mathematical concepts as legitimate objects of philosophical study, we can also justify the inclusion of mathematical theories. Finally, I show why a linguistic analysis of the content and structure of what we say cannot suffice to justify the inclusion of either physical concepts or physical theories.

While semantic philosophers of mathematics have come to rely on the syntax of logic or model theory to provide justification for the inclusion of mathematical concepts and theories, the linguistic approach relies on category-theoretic notions. If we restrict our analysis to what can be said in, and about, mathematical theories, then we can justify the inclusion of mathematical concepts and theories by representing talk of their content and structure in category-theoretic terms. What category theory does, as far as our talk of mathematical concepts and relations is concerned, is provide a means for organizing and classifying what we say about 'the structure of the relationship' between various mathematical concepts in various mathematical theories. We say that category theory is the language of mathematical concepts and relations because it allows us to talk about their structure in terms of "objects" and "arrows", wherein such terms are taken as syntactic assemblages waiting for an interpretation of the appropriate sort to give them formulas meaning.

Likewise, at the level of mathematical theories themselves, our talk of 'the structure of the relationship' between mathematical theories and their relations is represented by category-theoretic notions. We say that category theory is the language of mathematical theories and their relations because it allows us to talk about their structure in terms of "objects" and "functors", wherein such terms are, again, taken as syntactic assemblages waiting for an interpretation of the appropriate sort to give them formulas meaning. Thus, recalling our semantic aim (to justify the inclusion of concepts by offering an analysis of the content and structure of what we say), it appears that both mathematical concepts and mathematical
theories can, when writ in the language of category theory, be taken as legitimate objects of philosophical study.

Returning to our initial questions, we note that while the theoretical axioms, definitions, etc., may be conventions, i.e., may depend on the terms, or model, chosen, the category itself remains as an objective representation of both the content and structure of what such axioms and definitions say about the corresponding concepts from within a given mathematical theory. Conventionalism, then, is a consequence of accepting that mathematical theories themselves are local domains of mathematical discourse, or in Carnapian terms are local "linguistic frameworks". It is not a condition of the description.

The reason why we do not run into the problems that Carnap encountered, however, is that we can describe the structure of theories themselves in linguistic terms, i.e., in category theoretic-terms. So when Tait claims that Carnap is right that 'external questions' of existence have no prima-facie sense. But ... his notion of a linguistic framework fails [because] linguistic frameworks are constructed in our everyday language; and it is hard to see how we can convincingly determine when we have a 'good' framework and when we do not. (Tait in Hart, [1996], p. 151)

what we must note is that there is a middle ground. If we take mathematical theories to be linguistically represented by categories, then the category-theoretic version of a linguistic framework does not fail. We can objectively determine when we have a 'good' framework and when we do not. There is no need, therefore, to reduce the content or structure of mathematical theories to 'atoms of meaning'; we get all the meaning we want or need from within mathematical theories.

The question that remains to be considered is: Why is a linguistic analysis of the content and structure of what we say not sufficient to justify the inclusion of either physical concepts or physical theories? What we first note is that our "facts" occur at different levels. Mathematical facts occur within theories; they are fixed by the way things are in a given mathematical theory. Insofar as mathematical theories and their elements and relations are themselves linguistic entities, then so are mathematical facts. Physical facts, on the other hand, occur in reality, their constituents are ontological entities. The reference of a physical concept, i.e., a physical object, is not a linguistic entity, it is an ontological one. The question of the reality of physical objects and the truth of physical propositions thus cannot be settled linguistically: it must crucially depend on some extra-linguistic process whereby the connection between language and reality is both established and justified. This is the lacuna that naturalized epistemology fills.

What the naturalized epistemologist and the platonist must realize, however, is that this demand for a non-linguistic experience (causal or intuitive) does not apply to mathematical knowledge. Quite simply there is no gap in mathematical discourse between language and reality. Mathematical reality in contained within, as opposed to merely constrained by, the language of mathematical discourse. Insofar as the constituents of mathematical propositions and the referents of mathematical concepts occur within local domains of mathematical discourse, mathematical facts are in no need of assistance from either physical or metaphysical facts. There is therefore no need to appeal to the ways things are in reality (metaphysical or physical) to justify the inclusion of mathematical concepts. Likewise, then, our knowledge of mathematical facts is in no need of assistance from any extra-linguistic experience.
This distinction, between mathematics as a descriptive science (the view that mathematics is about mathematical objects) and mathematics as a descriptive discourse (the view that mathematics talks about mathematical objects), itself implies a difference in experience. To experience a mathematical fact we require a meet between use, proof and language; whereas, to experience a physical fact we require a mesh between reality, observation and language. If, then, in our account of mathematical experience, there is to be any appeal to mathematical intuition it must be one which distinguishes it from both empirical and pure intuition in the following manner:

What we call "mathematical intuition" . . . is not a criterion for correct usage. Rather having mastered the usage, we develop feeling, schematic pictures, etc., which guide us . . . What is objective about the existence of mathematical objects is that we speak a common language about them and this includes our agreement about what counts as warrant for what we say. (Tait in Hart, [1996], p. 147 & 149)

3. Semantic Realism

So where does this analysis of discourse leave our mathematician? He is a modified-formalist, or structuralist, to the extent that he believes that a linguistic analysis of mathematical structure justifies our accepting mathematical concepts and theories as legitimate objects of philosophical study. He is an internal realist to the extent that he believes that a linguistic analysis of the way things are within a given theory justifies our accepting that mathematical statements meaningfully talk about mathematical objects. Thus, what the mathematician qua semantic realist believes is that meaning, truth and existence are a linguistic notions, as opposed to a logical, psychological or metaphysical notions. This is why the mathematician means what he says.

References


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