

Two-Way ANOVA Source Table

In the Two-Way (Factorial) ANOVA, the two factors represent separate sources of variance. The interaction of these two factors also presents an independent source of variation. Suppose a design in which a Factor A has two levels (e.g., Treatment vs. Control; Male vs. Female) and Factor B has three levels (e.g., High, Medium, and Low ability groups; Control, Treatment 1, and Treatment 2). The design has 2x3=6 cell (group) means.

| | | | | | | | | | |
|----------------|----------------|----------------|----------------|----------------|---------------------------|----------------|----------------|----------------|-----|
| | B ₁ | B ₂ | B ₃ | | From the board example | B ₁ | B ₂ | B ₃ | |
| A ₁ | \bar{Y}_{11} | \bar{Y}_{12} | \bar{Y}_{13} | \bar{Y}_{1*} | A ₁ | 6 | 10 | 11 | 9 |
| A ₂ | \bar{Y}_{21} | \bar{Y}_{22} | \bar{Y}_{23} | \bar{Y}_{2*} | A ₂ | 6 | 6 | 12 | 8 |
| | \bar{Y}_{*1} | \bar{Y}_{*2} | \bar{Y}_{*3} | \bar{Y}_{**} | | 6 | 8 | 11.5 | 8.5 |

ANOVA MODEL: $Y_{ijk} = \mu_{**} + \alpha_j + \beta_k + \alpha\beta_{jk} + \epsilon_{ijk}$

Factor A has two marginal means, \bar{Y}_{1*} and \bar{Y}_{2*} , and ($A-1=2-1=1$) degree of freedom. The null hypothesis for Factor A is $H_0: \mu_{1*} = \mu_{2*}$ or $H_0: \sum \alpha_j^2 = 0$.

Factor B has three marginal means, \bar{Y}_{*1} , \bar{Y}_{*2} , and \bar{Y}_{*3} , and ($B-1=3-1=2$) degrees of freedom. The null hypothesis for Factor B is $H_0: \mu_{*1} = \mu_{*2} = \mu_{*3}$ or $H_0: \sum \beta_k^2 = 0$.

The interaction term is multiplicative conceptually; thus, the AxB interaction has ($A-1$)x($B-1$) degrees of freedom. The null hypothesis quite complex,

$$H_0: (\mu_{11} - \mu_{21}) = (\mu_{12} - \mu_{22}) = (\mu_{13} - \mu_{23}) \text{ or } H_0: \sum \alpha\beta_{jk}^2 = 0.$$

It implies that the absence of an interaction indicates that the main effects of Factor A are independent of the main effects of Factor B. Since interactions are symmetric, the absence of an interaction also indicates that the main effects of Factor B are independent of the main effects of Factor A.

The computation of the Sums of Squares (SS) for the main effects of Factors A and B are similar to the one-way analysis. Marginal means are subtracted from the grand mean, squared, weighted by the marginal sample size, and summed. The interaction SS involve subtracting the cell means from the grand mean and also subtracting the main effects of A and B, squaring, weighting by cell sample size, and summing over all cells. There is an algebraic simplification of this formula.

| Source | Sum of Squares | df | Mean Squares | F |
|---|---|--------------------------|----------------------|------------------|
| Main Effects | | | | |
| Factor A ($\sum \alpha_j^2$) | $\sum n_A (\bar{Y}_{A*} - \bar{Y}_{**})^2$ | $A - 1$ | $SS_A / A - 1$ | MS_A / MS_W |
| Factor B ($\sum \beta_k^2$) | $\sum n_B (\bar{Y}_{*B} - \bar{Y}_{**})^2$ | $B - 1$ | $SS_B / B - 1$ | MS_B / MS_W |
| Interaction AxB | | | | |
| ($\sum \alpha\beta_{jk}^2$) | $\sum n_{AB} (\bar{Y}_{AB} - \bar{Y}_{A*} - \bar{Y}_{*B} + \bar{Y}_{**})^2$ | $(A - 1) \times (B - 1)$ | SS_{AB} / df_{AB} | MS_{AB} / MS_W |
| Within Groups ($\sum \epsilon_{ijk}^2$) (Error Variance) | $\sum n_{AB} (Y_i - \bar{Y}_{AB})^2$ | $N - AB$ | SS_W / df_W | |
| Total Variance | $\sum (Y_i - \bar{Y}_{**})^2$ | $N - 1$ | $s^2 = SS_T / N - 1$ | |

where, N = total number of cases, A = number of groups for Factor A, B = number of groups for Factor B, \bar{Y}_{**} = the grand mean of Y across all groups. Y_i = each individual score on Y , n_A = the number of cases in each group of Factor A, n_B = the number of cases in each group of Factor B, and n_{AB} = the number of cases in AB cell.

Treatment magnitude for each effect can be calculated for each effect. For example, the eta-squared for Factor B is $R^2 = \eta^2 = SS_B / SS_T$.

Post-Hoc Analysis in Factorial Designs

In the absence of a significant AxB Interaction the Post Hoc methods used in the one-way ANOVA can be applied to the marginal means of a significant main effect. Based on the previous example, if the AxB Interaction does not reach statistical significance, but the main effect of Factor A is statistically significant, Tukey's HSD procedure can be applied to the marginal means of A, \bar{Y}_{1*} and \bar{Y}_{2*} .

If the AxB Interaction effect is statistically significant, the most common method for Post Hoc analysis is called **Simple Main Effects**. This involves testing for significant differences of one Factor at each level of the other factor. From the previous example, we could test the Simple Main Effect of B at A_1 . This would involve the comparison of \bar{Y}_{11} , \bar{Y}_{12} , and \bar{Y}_{13} . The Sums of Squares for B at A_1 is calculated using the marginal mean for A_1 , \bar{Y}_{1*} . $(SS_{A_B \text{ at } A_1}) = \sum n_{A_1} (\bar{Y}_{1B} - \bar{Y}_{1*})^2$. The Error Term (MS_w) from the original source table would be used to form an F -ratio with $B-1=2$ and df_w degrees of freedom. This would also be computed for the other level of A. In this case, if a statistically significant difference is found post-hoc methods are still needed because there are more than two groups.

Therefore, it may be more efficient to test Simple Main Effects of A at each level of B. That is, testing A at B_1 indicates whether \bar{Y}_{11} is significantly different from \bar{Y}_{21} . Testing A at B_2 indicates whether \bar{Y}_{12} is significantly different from \bar{Y}_{22} . Testing A at B_3 indicates whether \bar{Y}_{13} is significantly different from \bar{Y}_{23} .

| Source | SS | df | MS | F | η^2 |
|-----------------------------------|-----|----|-------|---------|----------|
| Main Effects | | | | | |
| Factor A | 6 | 1 | 6 | 1.227 | .024 |
| Factor B | 124 | 2 | 62 | 12.682* | .504 |
| Interaction AxB | 28 | 2 | 14 | 2.864* | .114 |
| Simple Main Effects | | | | | |
| A at B_1 | 0 | 1 | 0 | 0.000 | |
| A at B_2 | 32 | 1 | 32 | 6.545* | |
| A at B_3 | 2 | 1 | 2 | 0.409 | |
| Within Groups (Error Variance) | 88 | 18 | 4.89 | | |
| Total Variance | 246 | 23 | 10.70 | | |

* indicates significant at the $\alpha = .10$ level.

In the example presented, it appears that A_1 and A_2 are equivalent at levels B_1 and B_3 , therefore, the significant interaction occurs between A_1 and A_2 at B_2 , $F(1,18) = 6.545$.