```
data respx;
input   Y        X1        X2;
x1sq=x1*x1;x2sq=x2*x2;
yc=y-( -0.5411915);c1=x1-(-0.4665662);
c2=x2-0.5656137;c1sq=c1**2;c2sq=c2*c2; i1=c1-.58199;i1sq=i1**2;
cards;
  -1.740293 -1.262304 0.6635743
  -0.446665 -0.181167 0.1643075
      0.293 -0.721142 0.5733697
  0.0551676 -0.108962 0.5621188
  -0.955652 -2.113664 0.8103583
  -3.787614 -2.682232 0.5132363
  -1.815063 -0.707801 0.0900045
  -2.592468 -2.342706 0.9829417
  0.0967905 0.0309965 0.2489934
  1.4451384 1.1897634 0.5305004
  -2.067291 1.7290696 0.8743272
  -0.183809 -0.662325 0.6307918
  -1.877213 -1.042587 0.9060106
  2.3439291 0.2866271 0.7400729
  1.1119835  -0.17478 0.9767086
  -1.496547 1.6706253 0.3901691
  0.4293722 -1.117587 0.6464178
   -0.15269 -0.229022 0.2464271
  0.7480006 -0.788767 0.6758685
  -0.231906  -0.10336 0.0860751
;
proc reg data=respx;model y= x1 x2;output out=respx p=yhat r=e;run;
proc plot data=respx;plot e*yhat;run;
proc corr data=respx;
var y x1 x2 x1sq x2sq c1 c2 c1sq c2sq e;run;
proc glm data=respx;
model y = x1 x2 x1*x1 x2*x2/ solution;run;
proc reg data=respx;
model y = x1 x2 x1sq x2sq/ stb clb tol vif;run;
proc reg data=respx;
model y = c1 c2 c1sq c2sq/ stb clb tol vif;run;
proc reg data=respx;
model y = c1 c2 c1sq / stb clb tol vif;run;
```

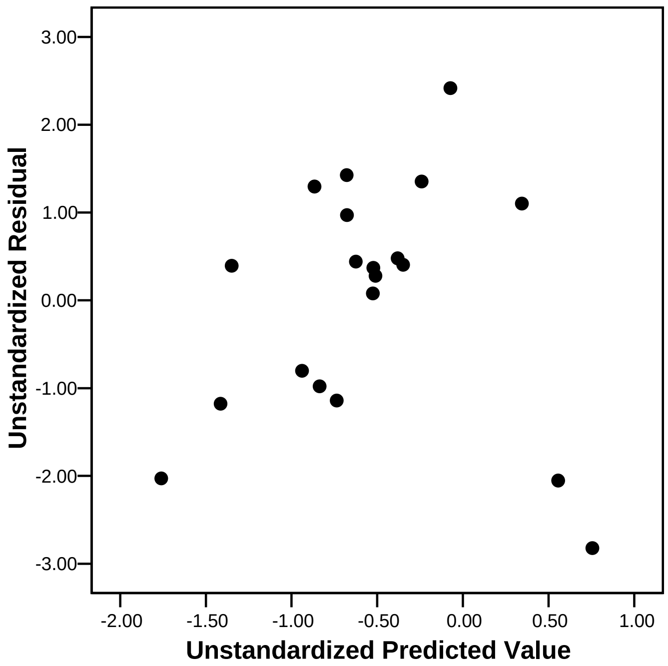
Analysis of Variance

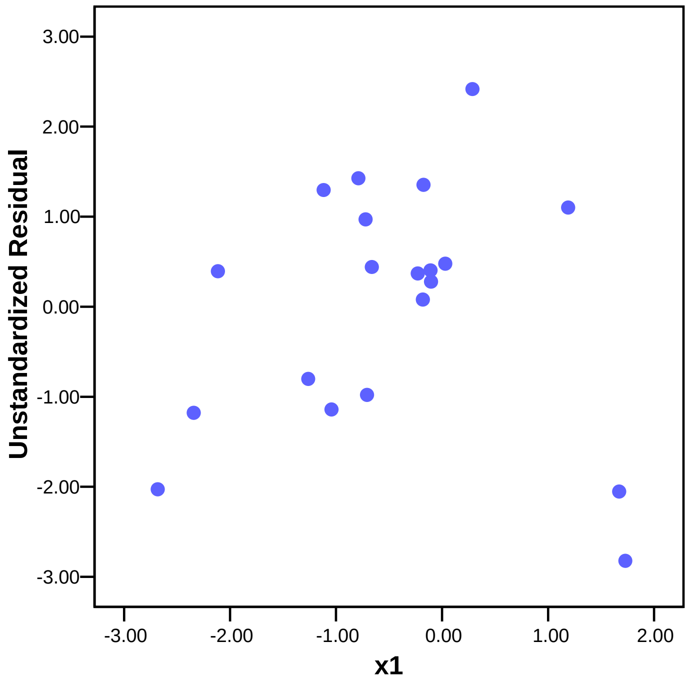
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	7.37345	3.68672	1.79	0.1976
Error	17	35.08077	2.06357		
Corrected Total	19	42.45422			

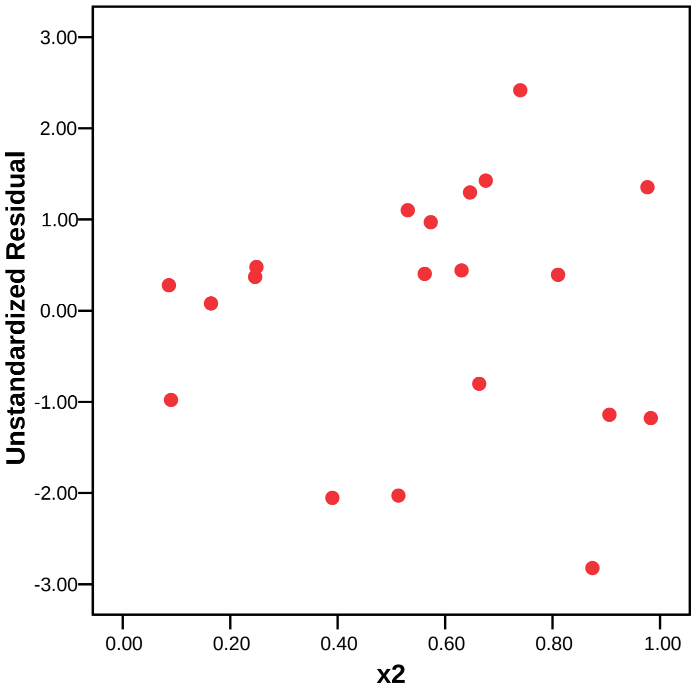
Root MSE	1.43651	R-Square	0.1737
Dependent Mean	-0.54119	Adj R-Sq	0.0765
Coeff Var	-265.43559		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	-0.48357	0.73102	-0.66	0.5172
X1	1	0.54184	0.28720	1.89	0.0764
X2	1	0.34508	1.18850	0.29	0.7751







Variable	N	Mean	Std Dev	Sum	Minimum	Maximum
Y	20	-0.54119	1.49480	-10.82383	-3.78761	2.34393
X1	20	-0.46657	1.17486	-9.33132	-2.68223	1.72907
X2	20	0.56561	0.28390	11.31227	0.08608	0.98294
x1sq	20	1.52896	2.05465	30.57917	0.0009608	7.19437
x2sq	20	0.39649	0.30885	7.92974	0.00741	0.96617
c1	20	-5E-9	1.17486	-1E-7	-2.21567	2.19564
c2	20	-2E-8	0.28390	-4E-7	-0.47954	0.41733
c1sq	20	1.31127	1.80984	26.22549	0.03832	4.90917
c2sq	20	0.07657	0.07958	1.53137	0.0000122	0.22996
e	20	0	1.35881	0	-2.82231	2.41681

	Y	X1	X2	x1sq	x2sq	c1	c2	c1sq	c2sq	e
Y	1.00000	0.41180	-0.02582	-0.71900	-0.07668	0.41180	-0.02582	-0.56681	-0.19343	**0.90902**
		0.0712	0.9140	0.0004	0.7480	0.0712	0.9140	0.0092	0.4139	<.0001
X1	0.41180	1.00000	-0.21452	**-0.47678**	-0.22546	1.00000	-0.21452	0.06447	-0.00933	**0.00000**
	0.0712		0.3638	0.0335	0.3392	<.0001	0.3638	0.7871	0.9689	1.0000
X2	-0.02582	-0.21452	1.00000	0.34937	**0.96884**	-0.21452	1.00000	0.26669	-0.27548	**0.00000**
	0.9140	0.3638		0.1311	<.0001	0.3638	<.0001	0.2557	0.2398	1.0000
x1sq	-0.71900	-0.47678	0.34937	1.00000	0.32741	-0.47678	0.34937	0.84646	-0.13924	**-0.59278**
	0.0004	0.0335	0.1311		0.1588	0.0335	0.1311	<.0001	0.5582	0.0059
x2sq	-0.07668	-0.22546	0.96884	0.32741	1.00000	-0.22546	0.96884	0.23512	-0.02879	-0.04858
	0.7480	0.3392	<.0001	0.1588		0.3392	<.0001	0.3183	0.9041	0.8388
c1	0.41180	1.00000	-0.21452	-0.47678	-0.22546	1.00000	-0.21452	**0.06447**	-0.00933	0.00000
	0.0712	<.0001	0.3638	0.0335	0.3392		0.3638	0.7871	0.9689	1.0000
c2	-0.02582	-0.21452	1.00000	0.34937	0.96884	-0.21452	1.00000	0.26669	**-0.27548**	0.00000
	0.9140	0.3638	<.0001	0.1311	<.0001	0.3638		0.2557	0.2398	1.0000
c1sq	-0.56681	0.06447	0.26669	0.84646	0.23512	0.06447	0.26669	1.00000	-0.16373	-0.67297
	0.0092	0.7871	0.2557	<.0001	0.3183	0.7871	0.2557		0.4904	0.0011
c2sq	-0.19343	-0.00933	-0.27548	-0.13924	-0.02879	-0.00933	-0.27548	-0.16373	1.00000	-0.18856
	0.4139	0.9689	0.2398	0.5582	0.9041	0.9689	0.2398	0.4904		0.4260
e	0.90902	0.00000	0.00000	-0.59278	-0.04858	0.00000	0.00000	-0.67297	-0.18856	1.00000
	<.0001	1.0000	1.0000	0.0059	0.8388	1.0000	1.0000	0.0011	0.4260	

The GLM Procedure

```
proc glm data=respx;
model y = x1 x2 x1*x1 x2*x2/ solution;run;
```

Dependent Variable: Y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	27.14614315	6.78653579	6.65	0.0028
Error	15	15.30807317	1.02053821		
Corrected Total	19	42.45421631			

R-Square	Coeff Var	Root MSE	Y Mean
0.639422	-186.6653	1.010217	-0.541191

Source	DF	Type I SS	Mean Square	F Value	Pr > F
X1	1	7.19948464	7.19948464	7.05	0.0180
X2	1	0.17396503	0.17396503	0.17	0.6855
X1*X1	1	17.39467565	17.39467565	17.04	0.0009
X2*X2	1	2.37801782	2.37801782	2.33	0.1477

Source	DF	Type III SS	Mean Square	F Value	Pr > F
X1	1	0.19282742	0.19282742	0.19	0.6700
X2	1	3.63497100	3.63497100	3.56	0.0786
X1*X1	1	18.41583026	18.41583026	18.05	0.0007
X2*X2	1	2.37801782	2.37801782	2.33	0.1477

Parameter	Estimate	Standard Error	t Value	Pr > |t|
Intercept	-1.340849425	0.79553989	-1.69	0.1126
X1	0.098253453	0.22603623	0.43	0.6700
X2	6.302806708	3.33962842	1.89	0.0786
X1*X1	-0.571458428	0.13452519	-4.25	0.0007
X2*X2	-4.655175904	3.04960210	-1.53	0.1477

The REG Procedure

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	27.14614	6.78654	6.65	0.0028
Error	15	15.30807	1.02054		
Corrected Total	19	42.45422			

Root MSE	1.01022	R-Square	0.6394
Dependent Mean	-0.54119	Adj R-Sq	0.5433
Coeff Var	-186.66535		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Standardized Estimate	Tolerance	Variance Inflation
Intercept	1	-1.34085	0.79554	-1.69	0.1126	0	.	0
X1	**1**	**0.09825**	**0.22604**	**0.43**	**0.6700**	**0.07722**	**0.76164**	**1.31296**
X2	**1**	**6.30281**	**3.33963**	**1.89**	**0.0786**	**1.19705**	**0.05975**	**16.73580**
x1sq	1	-0.57146	0.13453	-4.25	0.0007	-0.78548	0.70306	1.42235
x2sq	1	-4.65518	3.04960	-1.53	0.1477	-0.96185	**0.06055**	**16.51655**

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	27.14614	6.78654	6.65	0.0028
Error	15	15.30807	1.02054		
Corrected Total	19	42.45422			

Root MSE	1.01022	R-Square	0.6394
Dependent Mean	-0.54119	Adj R-Sq	0.5433
Coeff Var	-186.66535		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Standardized Estimate	Tolerance	Variance Inflation
Intercept	1	0.56459	0.37947	1.49	0.1575	0	.	0
c1	**1**	**0.63150**	**0.20406**	**3.09**	**0.0074**	**0.49633**	**0.93454**	**1.07004**
c2	**1**	**1.03674**	**0.90157**	**1.15**	**0.2682**	**0.19690**	**0.81988**	**1.21969**
c1sq	1	-0.57146	0.13453	-4.25	0.0007	**-0.69190**	**0.90613**	**1.10360**
c2sq	1	-4.65518	3.04960	-1.53	0.1477	**-0.24784**	**0.91192**	**1.09658**

Analysis of Variance

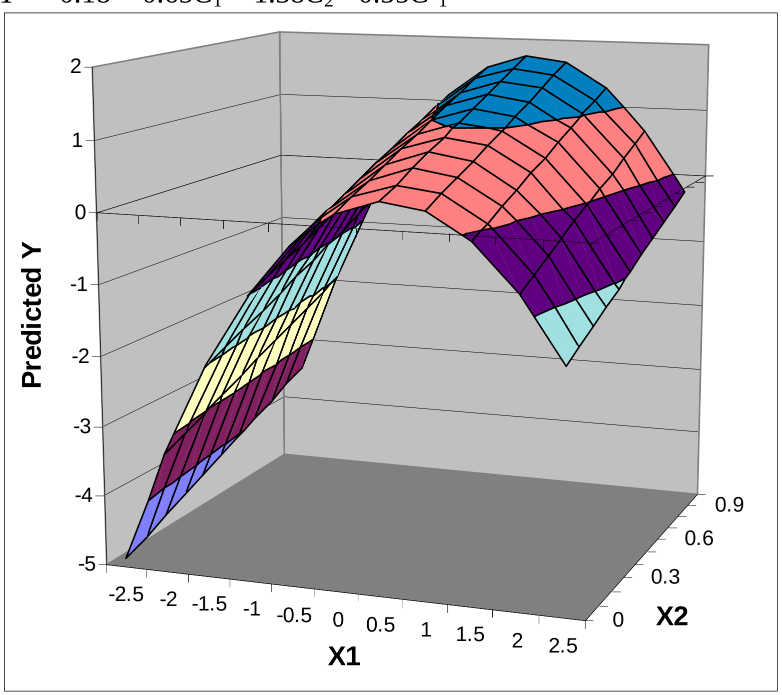
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	24.76813	8.25604	7.47	0.0024
Error	16	17.68609	1.10538		
Corrected Total	19	42.45422			

Root MSE	1.05137	R-Square	0.5834
Dependent Mean	-0.54119	Adj R-Sq	0.5053
Coeff Var	-194.26967		

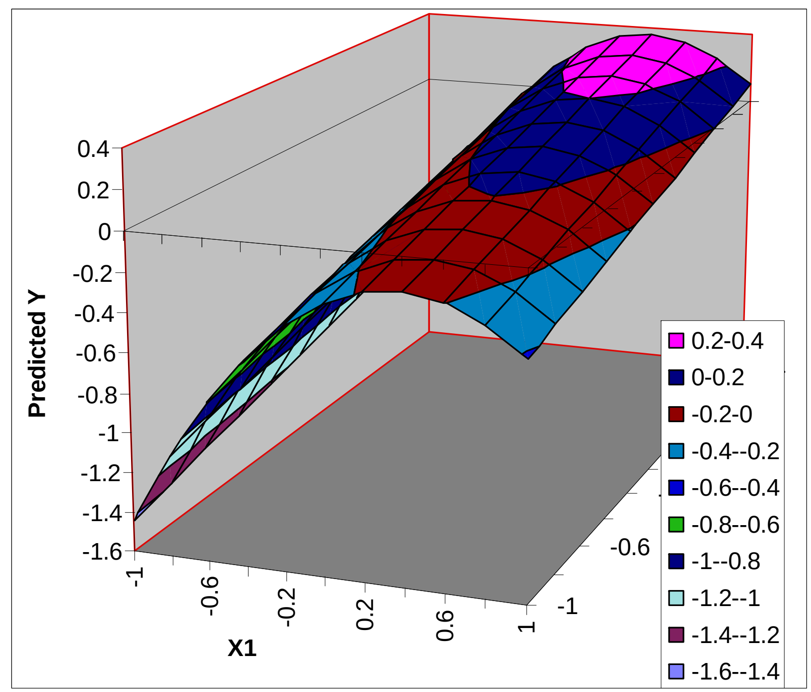
Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Standardized Estimate	Tolerance	Variance Inflation
Intercept	1	0.18419	0.29784	0.62	0.5450	0	.	0
c1	1	0.65053	0.21197	3.07	0.0073	0.51129	0.93804	1.06605
c2	1	1.38206	0.90828	1.52	0.1476	0.26249	0.87496	1.14290
c1sq	1	-0.55319	0.13945	-3.97	0.0011	-0.66977	0.91336	1.09486

$\hat{Y} = 0.18 + 0.65C_1 + 1.38C_2 - 0.55C^2_1$



$\hat{Z}_Y = 0.51Z_1 + 0.26Z_2 + 0.67Z^2_1$

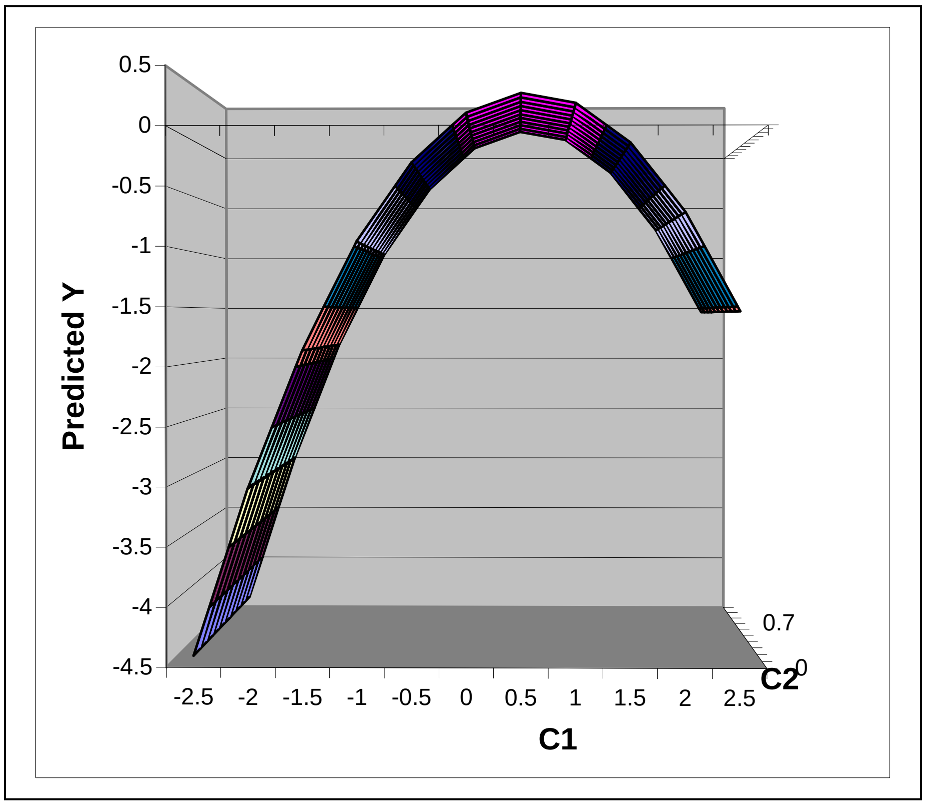


Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	22.20882	11.10441	9.32	0.0018
Error	17	20.24540	1.19091		
Corrected Total	19	42.45422			

Root MSE	1.09129	R-Square	0.5231
Dependent Mean	-0.54119	Adj R-Sq	0.4670

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Stand Estimate	Tolerance	Variance Inflation	95% Confidence Limits	
Intercept	1	**0.10411**	0.30428	0.34	0.7364	0	.	0	-0.53786	0.74608
c1	1	**0.57282**	0.21354	2.68	0.0157	0.45021	0.99584	1.00417	0.12229	1.02335
c1sq	1	-**0.49212**	0.13862	-3.55	0.0025	-0.59583	0.99584	1.00417	-0.78458	-0.19965



Even if the Coefficient for C1 (b_1 = .57282) had not been statistically significant, It would have been left in the model because it is the basis for the cross-product term (C1sq).

By taking the first derivative of the regression solution:

$\hat{Y} = b_0 + b_1 C + b_2 C^2$ with respect to C is: $\frac{d\hat{Y}}{dC} = b_1 + 2b_2 C$

Solving for zero finds the turning point: $C_{(\text{inf})} = \frac{-b_1}{2b_2}$

In this case: $C_{(\text{inf})}$ = -0.57282/[2(-0.49212)] = 0.58199.

Thus, if we center the data at the Turning Point, we have centered at a point where the curve changes direction.

There will a balance of positive negative values and there should be no linear association, only quadratic.

The output demonstrates this.

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	22.20882	11.10441	9.32	0.0018
Error	17	20.24540	1.19091		
Corrected Total	19	42.45422			

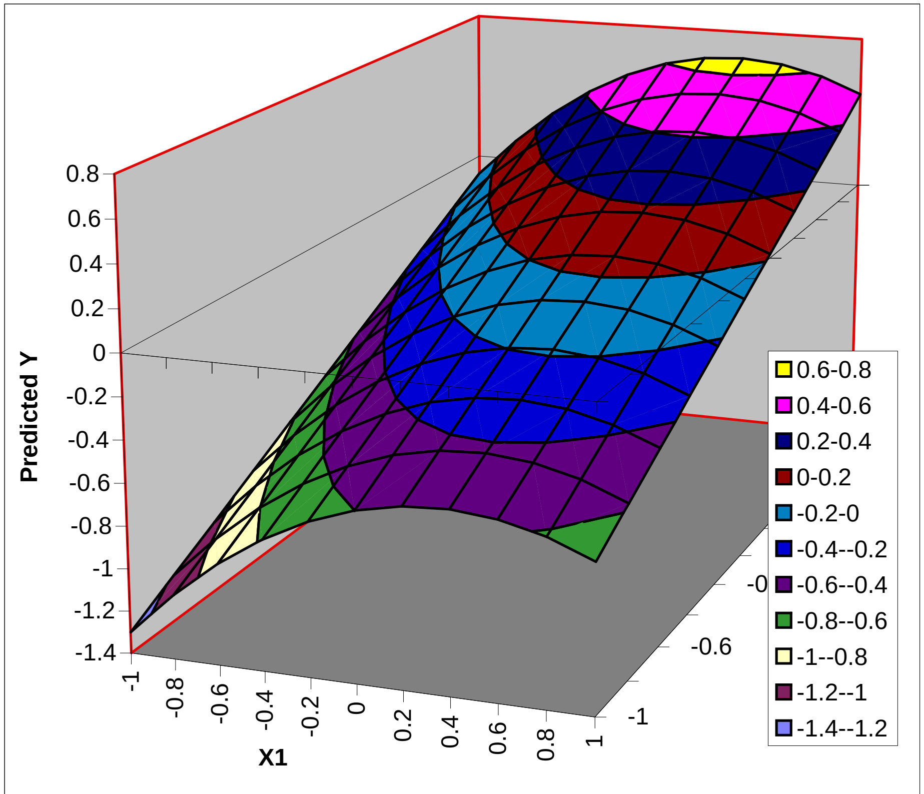
Root MSE	1.09129	R-Square	0.5231
Dependent Mean	-0.54119	Adj R-Sq	0.4670

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|	Tolerance	Variance Inflation	95% Confidence Limits	
Intercept	1	0.27080	0.30875	0.88	0.3927	.	0	-0.38061	0.92220
i1	1	**0.00000612**	0.25921	0.00	1.0000	0.67583	1.47966	-0.54689	0.54690
1sq	1	-0.49212	0.13862	-3.55	0.0025	0.67583	1.47966	-0.78458	-0.19965

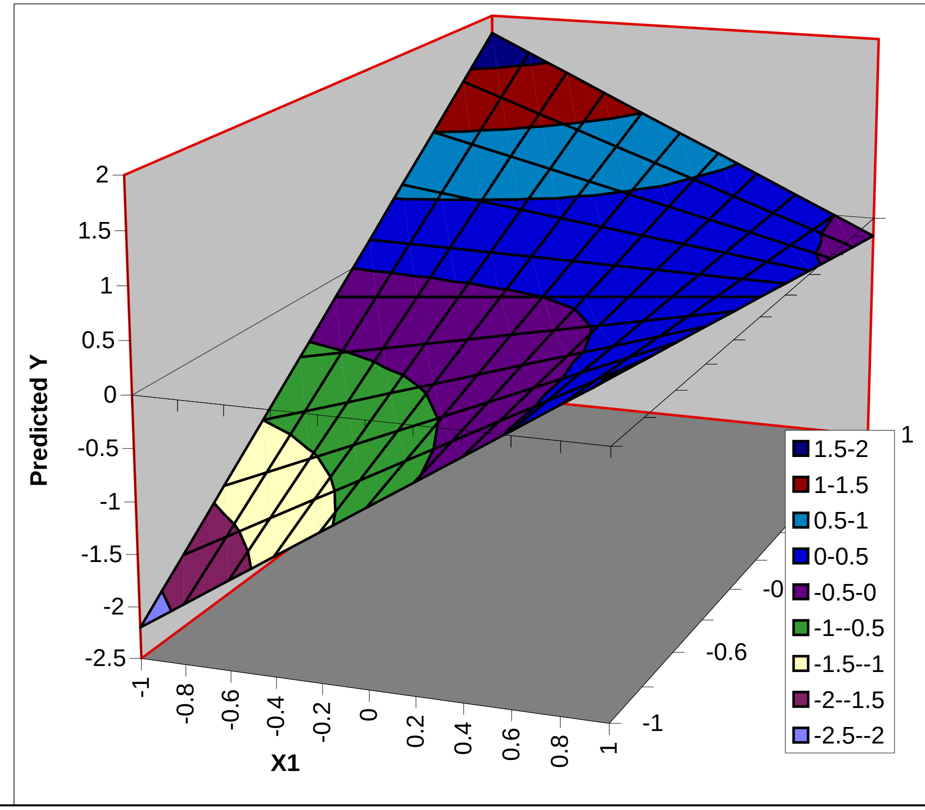
Thus, one can test whether the Turning Point in the data is different an *a priori* (Theoretically Meaningful) Turning Point by Subtracting the Theoretical Turning Point from X. From this if the regression coefficient for X is statistically significant then the Turning Point in the Observed Data is Significantly Different than the Hypothesized Turning Point.

One should note that this method can be highly influenced by Outliers.

$$Y = B_0 + B_1X_1 + B_2X_2 + B_3X_1^2$$



$$Y = B_0 + B_1X_1 + B_2X_2 + B_5X_1X_2$$



$\hat{Y} = .3(X_1) + .6(X_2) - .4(X_1^2)$

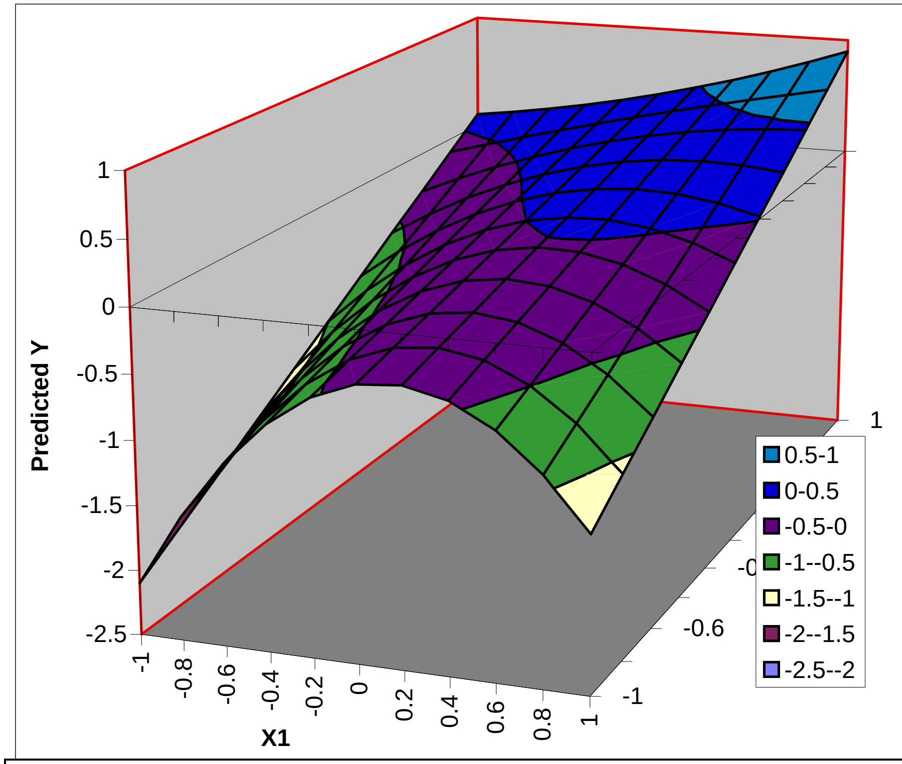
The X_2 relationship is linear (B_2=.6) The X_1 relationship has a linear component (.3), but has a negatively accelerating curve (-.4). There are no interactions.

$\hat{Y} = .2(X_1) + .8(X_2) - 1.2(X_1X_2)$

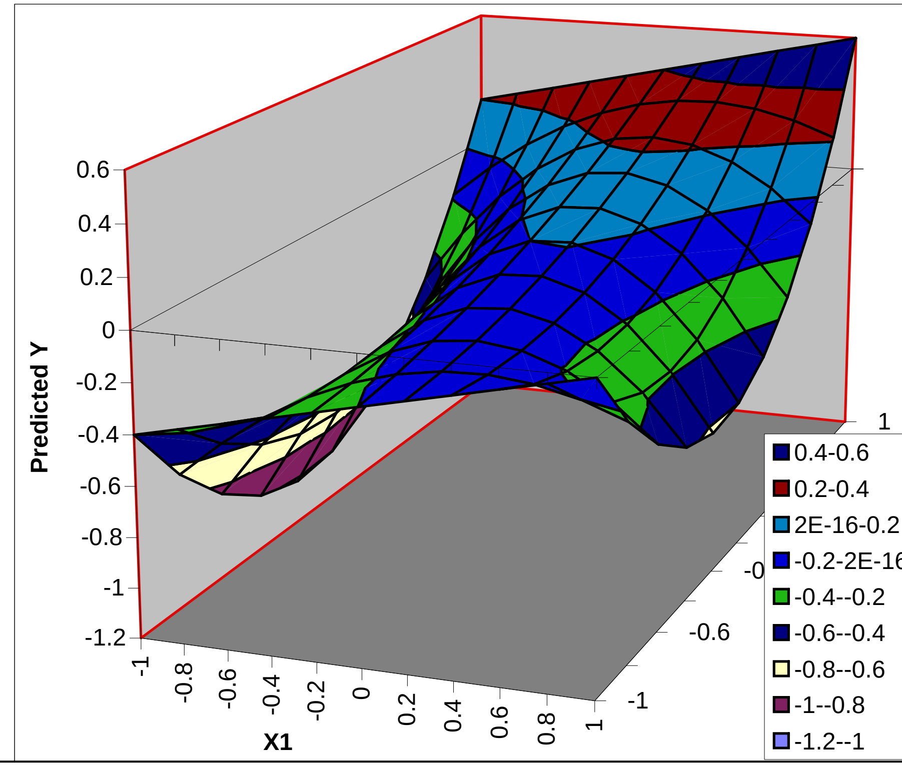
The X_1 and X_2 relationships are linear. The linear interaction (cross-product) shows that the linear relationship of X_1 with Y, changes as a linear function of X_2 . At X_2=0, the X_1 slope (B_1) = .2. For negative values of X_1, the X_1 slope becomes larger than .2. At X_2=.167, the X_1 slope (B_1) = 0. For X_2>.167, the X_1 slope (B_1) becomes negative.

Because interactions are symmetric, this can be described as the linear relationship of X_2 with Y changing as linear function of X_1. At X_1=0, the X_1 slope (B_1) = .8. For negative values of X_1, the X_1 slope > .8. At X_1=.667, the X_1 slope (B_1) = 0. For X_1>.667, the X_2 slope (B_2) becomes negative.

$$Y = B_0 + B_1X_1 + B_2X_2 + B_3X_1^2 + B_7X_1^2X_2$$



$$Y = B_0 + B_1X_1 + B_2X_2 + B_3X_1^2 + B_4X_2^2 + B_8X_1^2X_2^2$$



$\hat{Y} = .4(X_1) + .4(X_2) - .6(X_1^2) + .7(X_1^2X_2)$

The X_2 relationship is linear. The X_1 relationship has a linear component, but has a negatively accelerating curve (-.6).

The X_1-quadratic-by-X_2-linear interaction ($X_1^2X_2$) means that the quadratic relationship of X_1 with Y, changes as a linear function of X_2. As can be seen at negative values of X_2 , the relationship of X_1 with Y has a strong quadratic trend. At positive values of X_2 , the relationship of X_1 with Y is near zero.

Because interactions are symmetric, this can be described as the linear relationship of X_2 with Y changing as a quadratic function of X_1. At X_1=0, the X_2 slope (b_1) = .4. For both positive and negative values of X_1, the X_2 slope becomes larger than .4.

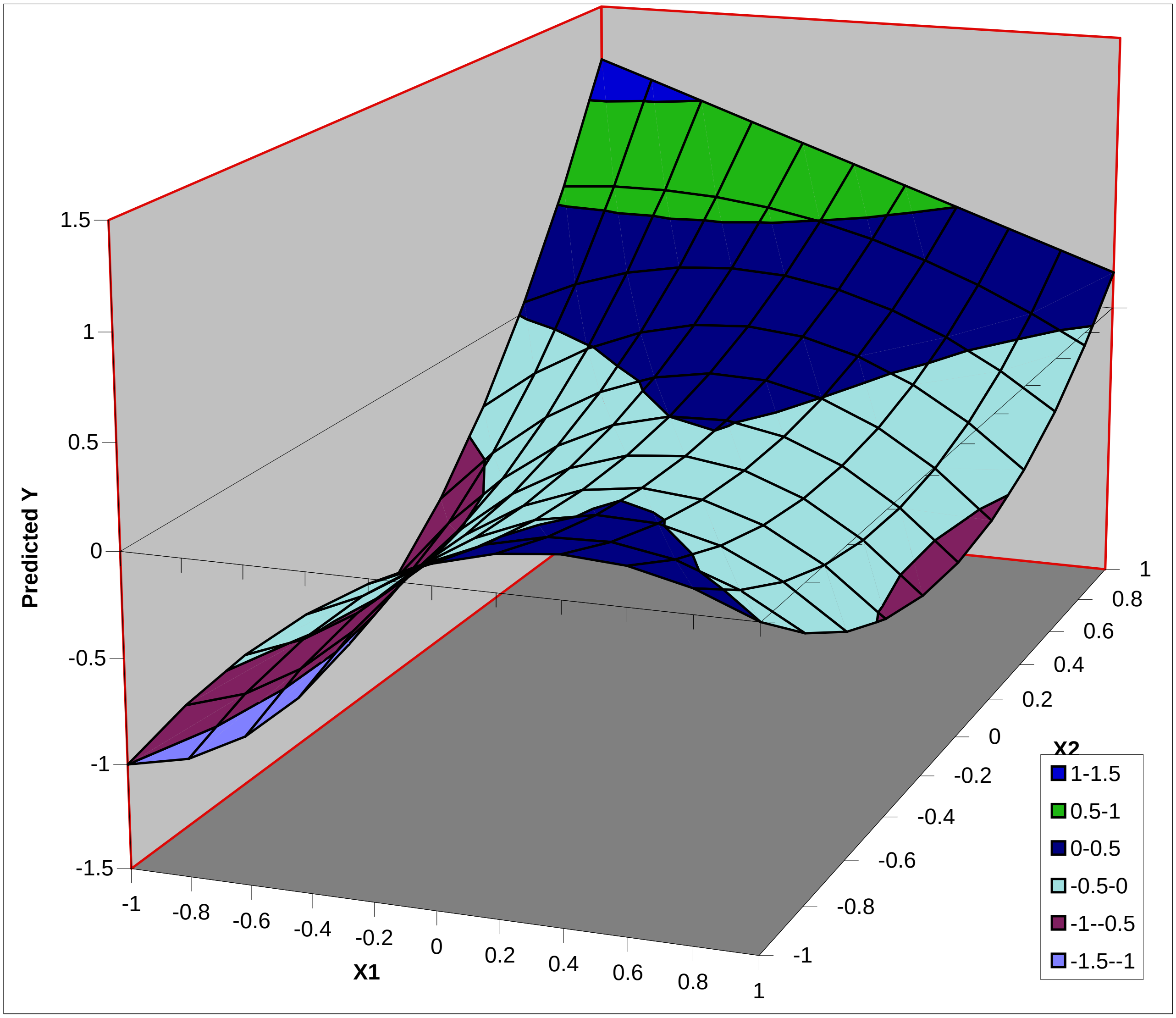
$\hat{Y} = .2(X_1) + .3(X_2) - .8(X_1^2) + .1(X_1^2) + .8(X_1^2X_2^2)$

The X_1 relationship has a linear component (.2), but has a negatively accelerating curve (-.8). The X_2 relationship has a linear component (.3), but has a slight positively accelerating curve (.1).

The X_1-quadratic-by-X_2-quadratic interaction ($X_1^2X_2^2$) means that the quadratic relationship of X_1 with Y, changes as a quadratic function of X_2. As can be seen, for more extreme positive and negative values of X_2 , the relationship of X_1 with Y has a slight linear trend. For less extreme values near X_2=0, the relationship of X_1 with Y has a strong quadratic trend.

Because interactions are symmetric, this can be described as the quadratic relationship of X_2 with Y changing as a quadratic function of X_1. At extreme positive and negative values of X_1, the X_2 trend is J-shaped. For values near X_1=0, the X_2 trend is linear.

$$Y = B_0 + B_1 X_1 + B_2 X_2 + B_3 X_1^2 + B_4 X_2^2 + B_5 X_1 X_2 + B_6 X_1 X_2^2 + B_7 X_1^2 X_2 + B_8 X_1^2 X_2^2$$



$$\hat{Y} = .2(X_1) + .3(X_2) - .8(X_1^2) + .4(X_2^2) - .5(X_1 X_2) - .2(X_1 X_2^2) + .3(X_1^2 X_2) + .5(X_1^2 X_2^2)$$